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Coincidence Wecken homotopies versus Wecken homotopies relative to a fixed homotopy in one of the maps

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ABSTRACT

We study the 1-parameter Wecken problem versus the restricted Wecken problem, for coincidence free pairs of maps between surfaces. For this we use properties of the function space between two surfaces and of the pure braid group on two strings of a surface. When the target surface is either the 2-sphere or the torus it is known that the two problems are the same. We classify most pairs of homotopy classes of maps according to the answer of the two problems are either the same or different when the target is either projective space or the Klein bottle. Some partial results are given for surfaces of negative Euler characteristic.

0. Introduction

A map $f : X \to X$ is said to be *minimal* if it achieves the least number of fixed points possible among all maps in its homotopy class. A homotopy $H : X \times I \to X$ is said to be *minimal* if for every $t \in I$ the map $H(\ ,t) : X \to X$ is minimal. The problem of finding *minimal* homotopies joining a given pair of minimal maps was initially proposed by H. Schirmer in her pioneering work [18]. This problem is also referred to as the 1-parameter problem for fixed points. A similar question for roots can be stated following exactly the same lines as in the fixed point case.

Let us consider the setting of coincidence between a pair of maps. Suppose that $f, g : X \to Y$ is a pair of maps between two spaces X, Y and that $H_1(\ ,\ ), H_2(\ ,\ ) : X \times I \to Y$ is a pair of homotopies. Analogous to the fixed point setting we have the following definitions.

Definition 1. The pair (f, g) is said to be *minimal* if it achieves the least number of coincidence points possible among all pairs (f', g') of maps in its homotopy class. This least number is called the minimal coincidence number for the homotopy class of the pair.

Definition 2. The pair of homotopies $F, G : X \times I \to Y$ is said to be minimal if for every $t \in I$ the pair of maps $F(\ ,t), G(\ ,t) : X \to Y$ is minimal.

There is more than one version of the 1-parameter problem for fixed points, in the case of coincidence theory. We will describe and study two such versions.

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Consider pairs $(f_1, g_1), (f_2, g_2)$ of minimal maps where $f_1, f_2 \in [f]$ and $g_1, g_2 \in [g]$. The first type of minimality under consideration is just the general coincidence problem where we ask if there is a homotopy between the two pairs such that the homotopy is minimal. The second type of minimality is when we choose one coordinate (for example the second) and select a homotopy G between g_1 and g_2 as part of the data. Then we may ask if there is a homotopy F connecting f_1 to f_2 such that the homotopy (F, G) is minimal. Of course if the answer of the first type of minimality question is negative, then the answer of the second type of minimality question is also negative. But when the answer of the first type of minimality question is positive, then we can question as to what happens with the second type of minimality problem for all possible choices of the homotopy G. We are primarily interested in deciding for which cases the answer of this question is yes (which in this case we say that *the two problems are the same*), and when this is not the case (which in this case we say that *the two problems are different*). The type of question above can in fact be addressed to a fixed pair of homotopy classes $([f], [g])$ and we study this in many cases.

As a summary let us state:

The general (or unrestricted) 1-parameter problem. Given two pairs (f_1, g_1), (f_2, g_2) of minimal maps which are homotopic, can one find a minimal homotopy (F, G) which connects these two pairs.

The restricted 1-parameter problem. Given two pairs (f_1, g_1),(f_2, g_2) of minimal maps which are homotopic and a fixed homotopy G between g_1 and g_2, can one find a minimal homotopy (F, G) which connects these two pairs.

Remarks.

1. A natural variation of the restricted 1-parameter problem is obtained by letting the first homotopy be fixed. Observe that the new problem is equivalent to the old problem above, in the sense that if we solve one of the problems for all maps then we solve the other for all maps. So we will not need to consider this new problem.
2. The restricted 1-parameter problem in the case where $g_1 = g_2 = Id_X$ and G is the constant homotopy equals identity, corresponds to the 1-parameter problem for fixed points.
3. The restricted 1-parameter problem in the case where $g_1 = g_2 = c$ (the constant map) and G is the constant homotopy equal to the constant map, corresponds to the 1-parameter problem for roots.
4. The motivation to study the restricted 1-parameter problem in general comes from the remarks 2 and 3 above.

In the papers [7–10] the authors study some aspects of the two problems stated above for maps between surfaces of nonnegative Euler characteristic. In the case where the target is either the 2-sphere or the torus, it has been established that the two problems are the same. For the former case see [7], Corollary 1.3, and for the latter case see [8], remark after Theorem 4.2. In the paper [8] a *minimal homotopy* is often referred to as a *Wecken homotopy*. We will adapt this naming convention here. We refer to Wecken homotopies for the restricted problem and Wecken homotopies for the unrestricted problem.

In this paper we study these two types of 1-parameter minimality problems for maps between surfaces under the assumption that minimal coincidence number is zero, i.e. we consider pairs (f, g) which are coincidence free. One consequence of the results obtained in this work is that the examples constructed in [17] of two homotopic maps f_1, f_2 which are fixed point free but cannot be joined by a fixed point free homotopy, also have the property that the pairs (f_1, id), (f_2, id) cannot be joined by a pair of coincidence free homotopies.

The main results of this paper are of two types. One is to show that for certain pairs of homotopy classes of maps $([f], [g])$ the two problems are equivalent. In certain cases the result is true for all possible pairs of homotopy classes, but this will depend on the spaces involved. The second type of results are some examples of pairs of homotopy classes of maps $([f], [g])$ such that one can find representative models where the two problems are not equivalent. We will see that the existence of such examples depends only on the homotopy class $[g]$, and not on the chosen model for the map $g \in [g]$, which is the second coordinate, as explained in Proposition 2.

The following are some of the main results that are obtained in this paper.

Theorem 3. *The two problems are equivalent if the domain is either S^2 or $\mathbb{R}P^2$ and the target is a closed surface with nonpositive Euler characteristic.*

For maps $(f, g) : M \to N$ where N has negative Euler characteristic, whenever the image of the homomorphism induced by the map on the fundamental group has infinite index, it is a free subgroup and of finite rank.

Theorem 4. *Let N be a closed surface with negative Euler characteristic, M a closed surface and let $g : M \to N$ be a map with the property that* $\mathrm{im}(g_\#)$, *the image of the fundamental group, is neither trivial nor isomorphic to Z. Then the general* 1*-parameter coincidence problem is equivalent to the restricted problem.*

Furthermore, we give a detailed analysis in the case that the target surface is the Klein bottle. One aspect of this analysis is to provide results in the other direction.

Examples. There are examples of a pair of maps from a closed surface into the Klein bottle where the two problems are not the same whenever the rank of the image of the homomorphism induced by the second map is either 0 or 1.

For maps (f, g) into $\mathbb{R}P^2$ we show:

Theorem 6. *For maps from a closed surface into $\mathbb{R}P^2$ the general problem and the restricted problem are related as follows*:

(I) *If* $\mathrm{Im}(f_\#, g_\#)$ *is trivial, then the two problems are equivalent.*
(II) *If* $\mathrm{Im}(f_\#, g_\#)$ *is* $Z_2 + 0$, *then the two problems are different if and only if the two homomorphisms* $(f_1, g)_\#, (f_2, g)_\#$ *are different. If* $\mathrm{Im}(f_\#, g_\#)$ *is either* $0 + Z_2$ *or the diagonal* $\Delta \subset Z_2 + Z_2$, *then the two problems are the same.*
(III) *If* $\mathrm{Im}(f_\#, g_\#)$ *is isomorphic to* $Z_2 + Z_2$, *the two problems are the same if and only if the two homomorphisms* $(f_1, g)_\#$ *and* $(f_2, g)_\#$ *are conjugate by an element of the subgroup* $\iota_\#^{-1}(Z_2 + 0)$.

This paper is divided into 5 sections besides the present one. In Section 1 we provide some preliminaries about the function spaces between two surfaces and some generalities about our main problem. See Theorem 1. In Section 2 we give a formulation for these problems in terms of surface braid groups. These result are given by Propositions 3 and 4. In Section 3 we provide the positive results. They are either relative a pair of homotopy classes of maps or general results about the spaces. The main results are Theorems 3 and 4 when the target is a surface of negative Euler characteristic, and Theorem 5 when the target is the Klein bottle. Section 4 provides examples where the two problems are not the same for certain pairs of homotopy classes of maps when the target is the Klein bottle. Moreover, an almost complete classification for the problem is given when also the domain is either the torus or the Klein bottle. In Section 5 we provide a complete classification for maps into $\mathbb{R}P^2$. The main result is Theorem 6.

1. Function space of maps between surfaces and some general results

Throughout the remainder of this work, we will assume that the chosen coordinate for the restricted problem is always the second coordinate. In this section we study quite general conditions which are sufficient to guarantee that the two types of minimality problems are the same. Here, the minimal number of coincidence points can be any nonnegative integer. We will show that the two problems are the same when certain conditions in terms of the fundamental group of the function spaces M^M, N^M and N^N are satisfied.

Let M and N be closed (compact without boundary) manifolds. Consider two pairs of minimal maps $(f_1, g_1), (f_2, g_2) : M \to N$ where f_1 is homotopic to f_2 and g_1 is homotopic to g_2. The next proposition is a parametrized weak version of the main result of [1].

Proposition 1. *Given two homotopies* G_1, G_2 *between* g_1 *and* g_2, *which are homotopic as paths in the function space* N^M, *relative to the end points* g_1, g_2, *then there is a Wecken homotopy* (F_1, G_1) *connecting the pairs* (f_1, g_1), (f_2, g_2) *if and only if there is a Wecken homotopy* (F_2, G_2) *connecting the pairs* (f_1, g_1), (f_2, g_2). *Furthermore,* $\mathrm{coin}(F_1, G_1) = \mathrm{coin}(F_2, G_2)$.

Proof. Consider the fiber pair $(N \times N, N \times N - \Delta) \to N$ where the map is the projection on the second coordinate. By hypothesis there is a homotopy $L : (M \times I) \times I \to N$ such that $L(\ , \ , 0) = G_1$ and $L(\ , \ , 1) = G_2$. Suppose that (F_1, G_1) is a Wecken homotopy for some homotopy F_1 and let C denote the set of coincidence points of (F_1, G_1) and $C_t = C \cap (M \times t)$. The map L has an initial lifting, which is (F_1, G_1), that sends the set $M \times I - C$ into $N \times N - \Delta$. Therefore there is a lifting $\hat{L}$ of L sending the set $(M \times I - C) \times I$ into $N \times N - \Delta$. This follows from the properties of fiber pairs, see [3]. At the level $s = 1$ the map $\hat{L}$ restricted to $M \times I \times 1$ is a pair of maps of the form F_2, G_2 because $\hat{L}$ covers L. Further, the subset $M \times I - C$ is mapped into $N \times N - \Delta$. This implies that the pair $\mathrm{coin}(F_2, G_2) \subset C$. Because (F_1, G_1) is a Wecken homotopy it follows that $\mathrm{coin}(F_2(\ , t), G_2(\ , t) \subset C_t$. From the minimality follows $\mathrm{coin}(F_2(\ , t), G_2(\ , t) = C_t$, hence (F_2, G_2) is also a Wecken homotopy and $\mathrm{coin}(F_2, G_2) = C$. Similarly, if we suppose that (F_2, G_2) is a Wecken homotopy we obtain F_1 such that (F_1, G_1) is a Wecken homotopy where $\mathrm{coin}(F_2, G_2) = \mathrm{coin}(F_1, G_1)$. The result then follows, including the furthermore part. □

The statement of our problem involves the choice of maps $g_1, g_2 \in [g]$. At least in the case of coincidence free maps, it is not difficult to show that the answer of our problem does not depend on the choice of the maps g_1, g_2. This is useful to study the problem independent if the answer is positive or negative. Likely this is also true if the minimal coincidence number is not zero.

Proposition 2. *Let* $([f], [g])$ *be a pair of homotopy classes such that the minimal coincidence number of the pair of classes is zero. Then there exist* (f_1, g_1), (f_2, g_2) *with* $f_i \in [f]$, $g_i \in [g]$, $i = 1, 2$, *coincidence free which can be connected by a Wecken homotopy* W_0 *for the unrestricted problem, but not by a Wecken homotopy for the restricted problem with a given homotopy* G *between* g_1 *and* g_2 *if and only if, for any* $g_1', g_2' \in [g]$ *one can find* $f_1', f_2' \in [f]$ *such that the two problems are not equivalent for* (f_1', g_1'), (f_2', g_2').

Proof. Given g_i' let G_i' be a homotopy between g_i and g_i', $i = 1, 2$. Let G_0 be the second coordinate of a Wecken homotopy for the unrestricted problem (such homotopy exists by hypothesis) between (f_1, g_1), (f_2, g_2). Using the lifting property of the fibration $S_2 \times S_2 - \Delta \to S_2$ by a standard argument we obtain pairs (f_i', g_i') which are coincidence free, and (f_i, g_i) and (f_i', g_i') can be joined by a Wecken homotopy W_i, $i = 1, 2$. Therefore (f_1', g_1'), (f_2', g_2') can be joined by a Wecken homotopy, which is the juxtaposition of $W_1^{-1} * W_0 * W_2$. Let $W_i = (F_i, G_i)$. We also get a homotopy between g_1' and g_2' (which is $G_1^{-1} * G * G_2$). Suppose by contradiction that we can join (f_1', g_1') to (f_2', g_2') by a Wecken homotopy for the restricted problem where the second coordinate is equal to $W' = G_1^{-1} * G * G_2$. Then we have a new Wecken homotopy $W_1 * W * W_2^{-1}$ between (f_1, g_1), (f_2, g_2) where the second coordinate is the homotopy $G_1 * G_1^{-1} * G * G_2 * G_2^{-1}$. But this homotopy, as a path, is homotopic to the path given by G. By Proposition 1 above the result follows. □

Now let us consider some special deformations.

Definition 3. We say that G is a special homotopy between g_1 and g_2 of type (I) if G is homotopic to the composite of the map $g_1 \times id : M \times I \to N \times I$ with an isotopy of the identity $G' : N \times I \to N$. We say that G is a special homotopy between g_1 and g_2 of type (II) if G is homotopic to the composite of an isotopy of the identity $G' : M \times I \to M$ with g_1.

Lemma 1. *If two pairs (f_1, g_1), (f_2, g_2) can be connected by a Wecken homotopy (F, G) where G is a special homotopy of a certain type, then they can be connected by a Wecken homotopy for any other choice of a special homotopy $\bar{G}$ between g_1 and g_2 of the same type.*

Proof. Let (F, G) be a Wecken homotopy where G is special and $\bar{G}$ be an arbitrary special homotopy between g_1 and g_2. So we have two cases to consider because of the definition of special homotopy. Let us consider the case where the special homotopy is of type (I). In this case $G' = G \circ (g_1 \times id)$ and $\bar{G}' = \bar{G} \circ (g_1 \times id)$. Define F_1 to be the composition $\bar{G}' \circ G'^{-1} \circ F$. It is straightforward to see that $(F_1, \bar{G})$ is a Wecken homotopy. The other case is similar and we leave to the reader. □

Given a map $g : M \to N$, let $g^* : N^N \to N^M$ and $g_* : M^M \to N^M$ be the induced maps obtained by composition on the right and on the left, respectively. We are interested in the restriction of these maps to the subspace Homeo(N) and Homeo(M), respectively, which by abuse of notation we will also use the same notation for the restriction. We will make clear whenever necessary the domains used. Let $(f_1, g_1), (f_2, g_2) : M \to N$ be two minimal pairs and G a fixed homotopy between g_1 and g_2. Consider the spaces Homeo(N), Homeo(M) with base points the identity maps, respectively.

Theorem 1. *If either of the induced homomorphisms g_{1*} or g_1^* on the fundamental group is onto, where the base points are id_M, id_N, g_1 in* Homeo(M), Homeo(N), N^M*, respectively, then the two pairs (f_1, g_1), (f_2, g_2) can be connected by a Wecken homotopy if and only if the pairs can be connected by a Wecken homotopy where the second homotopy is an arbitrary deformation G.*

Proof. The "if part" is clear. Let us prove the "only if part". First let us suppose that g_1^* is onto. Given a self-homotopy G of g by hypothesis we know that there is an isotopy G' such that the map $G' \circ (g \times id)$ regarded as a loop in the function space is homotopic to the loop defined by G. By Proposition 1 we know that there is a Wecken homotopy of $(F_1, G' \circ (g \times id))$ where the second homotopy is special. Now from Lemma 1 the result follows.

Now suppose that g_{1*} is onto. Given a self-homotopy G of g by hypothesis we know that there is an isotopy G' such that the map $g' \circ G'$ regarded as a loop in the function space is homotopic to the loop defined by G. By Proposition 1 we know that there is a Wecken homotopy of $(F_1, g \circ G')$ where the second homotopy is special. Now from Lemma 1 the result follows. □

Now we will describe some relevant results about the fundamental group of the various components of the space of maps between two closed surfaces. From [13] and [21], or [15], Theorems 1 and 2, we have:

Theorem 2. *Let N be an Eilenberg–MacLane space of type $(\pi, 1)$. Then the fundamental group $\pi_1(N^M, g)$, of the function space with base point a function $g : M \to N$, is given by the following*:

(a) *If π is abelian, then $\pi_1(N^M, g) = H^0(M, \pi)$.*
(b) *If π is not abelian, then $\pi_1(N^M, g) = C(\pi; g)$, where $C(\pi; g)$ is the centralizer of the image of the homomorphism induced by g in $\pi_1(N)$. Further each connected component of the space N^M is a $K(\pi, 1)$.*

It is worthy to notice that if $\pi_1(N)$ is abelian, then $\pi_1(N^M, g)$ depends neither on N nor on g. The elegant proof of the result above is obtained by showing that the evaluation map $N^M \ni f \to f(x_0)$ induces an isomorphism $\pi_1(N^M, g) \to C(\pi; g) \subset \pi_1(N, g(x_0))$. Before we continue we will describe the homomorphisms $(g^*)_\#$ and $(g_*)_\#$, i.e. the induced homomorphisms by the functions g^* and g_*, on the fundamental group, using the identification given by Theorem 2 with the fundamental group of the spaces. For g the identity, let $C(\pi; id)$ be denoted by $C(\pi)$ where is the center of the group π.

The following lemma will be useful to study our problem.

Lemma 2. *Let N be a $K(\pi, 1)$ and $g : M \to N$ a map.*

(a) *If π is non-abelian, then $\pi_1(N^N, id) = C(\pi)$, $\pi_1(N^M, g) = C(\pi; g)$ and the induced homomorphism $(g^*)_\# : C(\pi) \to C(\pi; g)$ is the inclusion.*
(b) *If π is abelian, then $\pi_1(N^N, id) = H^0(N, \pi)$, $\pi_1(N^M, g) = H^0(M, \pi)$ and the homomorphism $(g^*)_\# : H^0(N, \pi) \to H^0(M, \pi)$ is the induced map on the 0th-cohomology group by the map g, which is the identity.*

Consider $ev : M^M \to M$ the evaluation map and the induced homomorphism $ev_\# : \pi_1(M^M, id) \to \pi_1(M, x_0)$. The image of this homomorphism is the first Gottlieb group $G_1(M)$ or the Jiang subgroup of the identity $J(id_M)$, and it is contained in the center of $\pi_1(M, x_0)$. If we assume that M is also a $K(G, 1)$ then the analysis above is simpler, although more restrictive. We can state:

Lemma 3.

(a) *If π is non-abelian, then $\pi_1(N^M, g) = C(\pi; g)$. The induced homomorphism $(g_*)_\# : \pi_1(M^M, id) \to \pi_1(N^M, g)$ is the composite $\pi_1(M^M, id) \xrightarrow{ev_\#} G_1(M) \xrightarrow{g_\#} C(\pi; g)$.*
(b) *If π is abelian, then $\pi_1(N^M, g) = H^0(M, \pi) = \pi$. The induced homomorphism $(g_*)_\# : \pi_1(M^M, id) \to \pi_1(N^M, g)$ is given by the composite $\pi_1(M^M, id) \xrightarrow{ev_\#} \pi_1(M) \xrightarrow{g_\#} \pi$.*

The proofs of the two lemmas stated above are straightforward.

2. Braids and construction of examples

In this section we present an algebraic formulation of our problem using the pure braid groups on 2 strings of the target surface and the homomorphisms induced on the fundamental group of certain maps. These results will be most used to construct examples of pairs of maps where the two problems are not equivalent. Although it can be made more general, this procedure will be considered only in the case where the minimal number of coincidence is zero. Recall that the problem has been solved when the target is either the sphere S^2 or the torus T. In addition, the case where the target is the projective space will be treated in a different way in Section 5. So we will consider here only the setting where the target surface has Euler characteristic < 0 if it is orientable, and $\leqslant 0$ if it is nonorientable. For the procedure we use the pure braid group on 2 strings of the target surface. So we will recall some properties of these groups which are going to be used to formulate our algebraic version. Also because we will apply for the case where the target is the Klein bottle, we recall a presentation of this group which is suitable for our purpose.

In this section we consider two types of examples. For each type of example we present an algebraic version for the existence of the given type of example.

The first type of example under consideration is as follows: Construct pair of maps $(f_1, g), (f_2, g) : S_1 \to S_2$ which are coincidence free and that can be connected by a Wecken homotopy, but there is no Wecken homotopy if we impose that the second coordinate of the homotopy is the constant homotopy equal to g. If there exists such example, then of course the two problems are not equivalent. We now describe an algebraic formulation for the existence of such an example. A pair of maps $(f, g) : S_1 \to S_2$ which is coincidence free defines a map into the configuration space $(f, g) : S_1 \to S_2 \times S_2 - \Delta$. Let $(f_1, g), (f_2, g)$ be two pairs of coincidence free maps, then we have two maps $(f_1, g), (f_2, g) : S_1 \to S_2 \times S_2 - \Delta$. These two maps are homotopic if and only if there is a Wecken homotopy between the pairs $(f_1, g), (f_2, g)$. Since $S_2 \times S_2 - \Delta$ is a $K(\pi, 1)$ the two homomorphisms $(f_1, g)_\#, (f_2, g)_\#$ are conjugate (see [22], Chapter V, Section 4, Theorem 4.3). Therefore there exists an element $w \in P_2(S_2) = \pi_1(S_2 \times S_2 - \Delta)$ such that $(f_2, g)_\# = w(f_1, g)_\# w^{-1}$. When we project on the second coordinate we obtain a self-homotopy of g, which implies that $p_{2\#}(w)$ lies in the centralizer of $\mathrm{Im}(g_\#)$.

On the other hand the existence of the Wecken homotopy where the second coordinate is constant at g implies that $(f_2, g)_\# = v(f_1, g)_\# v^{-1}$, for some element $v \in F_1 \subset P_2(S_2)$, where F_1 is the set of classes of the pure braids with the second string constant. That is, $F_1 = \mathrm{Im}[\pi_1(N \setminus x_2, x_1) \to \pi_1(N \times N, N \times N \setminus \Delta; (x_1, x_2))] = \ker(p_{2\#})$. See [4] for more details.

We summarize in the following proposition.

Proposition 3. *The existence of such examples of the first type is equivalent to finding an element $w \in P_2(S_2) = \pi_1(S_2 \times S_2 - \Delta)$ such that $(f_2, g)_\# = w(f_1, g)_\# w^{-1}$, $p_{2\#}(w)$ lies in the centralizer of $\mathrm{Im}(g_\#)$, but there is no element $v \in F_1 \subset P_2(S_2)$ such that $(f_2, g)_\# = v(f_1, g)_\# v^{-1}$.*

We will construct examples of the first type making use of the proposition above. As pointed out before the proposition, the existence of such examples implies that the answer to our question is negative. We note that the converse is not true. That is, the nonexistence of such examples does not imply that our problem has a positive answer. For the converse it is not enough to only consider homotopies that are constant at g.

Now we describe how to construct the so called second type of examples. We give an equivalent algebraic version for the existence of an example of second type, which is indeed equivalent to our main problem. This algebraic formulation might have some interest in its own right. We would like to construct two pairs of coincidence free maps $(f_1, g), (f_2, g)$ which can be connected by a Wecken homotopy, but for some self-homotopy G of g there is not a Wecken homotopy between the two pairs such that the second coordinate of the homotopy is G.

Sometimes it is more convenient to use the following notation for the centralizer. If H is a subgroup of a group G denote by $C_G(H)$ the centralizer of H on G.

Proposition 4. *Suppose that two coincidence free pair of maps $(f_1, g), (f_2, g) : S_1 \to S_2$ can be connected by a Wecken homotopy. Then they can be connected by a Wecken homotopy where the second coordinate is an arbitrary self-homotopy of g if and only if the homomorphism $p_{2\#} : C_{\pi_1(S_2\times S_2-\Delta)}(\mathrm{Im}((f_1, g)_\#)) \to C_{\pi_1(S_2)}(\mathrm{Im}(g)_\#)$ is surjective.*

Proof. First let us show the "only if" part. Given $\alpha \in C_{\pi_1(S_2)}(\mathrm{Im}(g)_\#)$ consider a self-homotopy G of g which restricted to the base point cross the interval I is in the class of the loop α. By hypothesis there is a Wecken homotopy such that the second coordinate is the homotopy G. There is also a Wecken homotopy such that the second coordinate is the constant homotopy equals to g. Consider the composition of the first homotopy with the inverse of the other. It provides a self-homotopy of (f_1, g) whose restriction to the base point cross the unit interval provides the loop that projects to a loop on the class of α.

For the converse let $w \in P_2(S_2)$ be an element which has the property that $(f_2, g)_\# = w(f_1, g)_\# w^{-1}$. The projection of this element $p_{2\#}(w) = w'$ is an element of $C_{\pi_1(S_2)}(\mathrm{Im}(g_\#))$. Given any self-homotopy of g, it determines an element $v' \in C_{\pi_1(S_2)}(\mathrm{Im}(g_\#))$ which is the image of an element $v \in C_{\pi_1(S_2\times S_2-\Delta)}(\mathrm{Im}((f_1, g)_\#))$. Now consider the self-homotopy determined by vw^{-1} followed by the original homotopy determined by w and the result follows. □

This concludes our general analysis. Since the result above is going to be used for the case where the target is the Klein bottle, for the reader's convenience, we recall a presentation of the pure braid groups of the Klein bottle on two strings that we are going to use. One reference for (pure) braid groups which covers most of the surfaces is [19]. Here we use another presentation for the Klein bottle which is given in [11]. We use the convention for the product $\alpha\beta$ in π_1, which is the class of the loop α followed by β, which is the opposite convention in [19]. Let also the commutator of two elements a, b be denoted by $[a, b] = aba^{-1}b^{-1}$.

Let us consider the well known presentation of $\pi_1(K)$ given by $\langle \alpha, \beta \mid \alpha\beta\alpha\beta^{-1} = 1\rangle$, obtained by identifying the opposite sides of a square where two of them in the same direction and the other two on the opposite direction. Every element of $\pi_1(K)$ can be written uniquely in the form $\alpha^m\beta^n$ for m, n integers. So, using the projection surjective homomorphism $\pi_1(K - y_0) \to \pi_1(K)$, an arbitrary element of $\pi_1(K - y_0)$ can be written uniquely in the form $\alpha^m\beta^n\theta$, where θ is an element of the kernel of the homomorphism $\pi_1(K - y_0) \to \pi_1(K)$. So in particular, θ is a product of conjugates of B where $B = \alpha\beta\alpha\beta^{-1} \in \pi_1(K - y_0)$.

Given a base point $(x_1, x_2) \in K \times K - \Delta$ define α_i, β_i as the braids determined by the pair of loops where the jth-loop ($j \neq i$ for $i, j \in \{1, 2\}$) is the constant loop at the point x_j and the loops on the other coordinate are homotopic to the ones given by the α, β at base point x_i, respectively. The complete description of these elements can be found in Section 4 of [11].

For the purpose of our calculations it will be suitable to work with a presentation of $\pi_1(K \times K - \Delta)$ given in [11]. This presentation is:

A set of generators given by α_1, β_1, α_2, β_2 and B and the following relations:

$$\begin{aligned}
&\alpha_2\alpha_1\alpha_2^{-1} = B\alpha_1B^{-1},\\
&\alpha_2\beta_1\alpha_2^{-1} = B\alpha_1^{-1}\beta_1\alpha_1^{-1}B^{-1},\\
&\beta_2\alpha_1\beta_2^{-1} = \beta_1^{-1}\alpha_1^{-1}\beta_1,\\
&\beta_2\beta_1\beta_2^{-1} = \beta_1^{-1}B\beta_1^2,\\
&\alpha_2B\alpha_2^{-1} = B\alpha_1^{-1}B\alpha_1B^{-1},\\
&\beta_2B\beta_2^{-1} = \beta_1^{-1}B^{-1}\beta_1,\\
&\alpha_2^{-1}\alpha_1\alpha_2 = \alpha_1B^{-1}\alpha_1B\alpha_1^{-1},\\
&\alpha_2^{-1}\beta_1\alpha_2 = \alpha_1B^{-1}\alpha_1^{-1}B\beta_1\alpha_1B\alpha_1^{-1},\\
&\beta_2^{-1}\alpha_1\beta_2 = \alpha_1B^{-1},\\
&\beta_2^{-1}\beta_1\beta_2 = B\beta_1,\\
&\alpha_2^{-1}B\alpha_2 = \alpha_1B\alpha_1^{-1},\\
&\beta_2^{-1}B\beta_2 = B\beta_1B^{-1}\beta_1^{-1}B^{-1},\\
&B = \alpha_1\beta_1\alpha_1\beta_1^{-1} = \alpha_2\beta_2\alpha_2\beta_2^{-1}.
\end{aligned}$$

3. The equivalence of the two problems when $\chi(S_2) \leqslant 0$

In this section we present the main results as to when the two problems are equivalent. We begin by considering the case where the domain is either the sphere S^2 or the projective plane $\mathbb{R}P^2$. Then in Section 3.1 we will consider the case where the target surface has negative Euler characteristic. We will show that under certain hypothesis the two problems are the same. Finally in Section 3.2 we will consider the case where the target is the Klein bottle. Recall that if the target is the torus it is already known that the two problems are equivalent.

We are given two coincidence free pairs of maps (f_1, g), (f_2, g) which we assume can be joined by a Wecken homotopy and also an arbitrary homotopy G of g, and ask if they can be joined by a Wecken homotopy for which the second coordinate is G. We call this *the restricted problem with respect to the second coordinate.* In particular we focus on the situation where G is to be the constant homotopy at g. For the spaces under consideration in this paper the fundamental group of the function space with base point g will be trivial. So by Theorem 1 of Section 1, the results will follow for the general homotopy G.

We begin our consideration for maps from either S^2 or $\mathbb{R}P^2$.

Theorem 3. *The two problems are equivalent if the domain is either S^2 or $\mathbb{R}P^2$ and the target is a closed surface with nonpositive Euler characteristic.*

Proof. Suppose the domain is the sphere S^2. Any map is null homotopic and from Proposition 2 we can consider g to be the constant map. The image of the induced homomorphism by (f, g) is trivial and by Proposition 4 the result follows.

If the domain is the projective plane, we have that any map $f : \mathbb{R}P^2 \to N_g$, where N_g is the nonorientable closed surface of genus $\geqslant 2$, induces the trivial homomorphism on the fundamental group, because $\pi_1(N_g)$ is torsion free. The remainder of the proof is completely similar to the previous case and we leave it to the reader. $\square$

3.1. The case where $\chi(N) < 0$

The first result of this subsection is:

Theorem 4. *Let N be a closed surface with negative Euler characteristic, M a closed surface and let $g : M \to N$ be a map with the property that* $\mathrm{im}(g_\#)$*, the image of the fundamental group, is neither trivial nor isomorphic to Z. Then the general* 1*-parameter coincidence problem is equivalent to the restricted problem.*

Proof. It is well known that a subgroup of finite index of a surface group is again a surface group. If the subgroup is of infinite index then it is free. Because $g_\#(\pi_1(M))$ is neither trivial or isomorphic to Z, it follows that the centralizer of this group in $\pi_1(N)$ is trivial (in fact the component of the function space N^M which contains the element g is contractible as a result of Gottlieb [14, Lemma 2]). Therefore, from Proposition 1 in Section 1 the result follows.

Corollary 1. *The examples given in* [17]*, where M is a surface with negative Euler characteristic, cannot be connected by a Wecken homotopy regarded as coincidence problem.*

For maps such that the image of the induced homomorphism on the fundamental group is either trivial or isomorphic to Z we will show in the next section that there are examples where the two problems are not the same.

3.2. The case where the target is the Klein bottle

In this subsection we consider maps where the target is the Klein bottle K. Recall that the case where the target is the torus T, by [8], the two problems are equivalent. So this subsection completes the analysis of the setting where the Euler characteristic of the target is zero.

For maps from the Klein bottle or torus into the Klein bottle, from [10] or [9], respectively, we know that the two problems are not equivalent in general. But they are equivalent under certain hypotheses. We explore in this subsection both possibilities.

For a map $h : S \to K$ we will consider four cases depending on $\mathrm{Im}(h_\#)$. Case 0 is when $\mathrm{Im}(g_\#)$ is trivial. Case 1 is when $\mathrm{Im}(g_\#) \approx Z$. Case 2 is when $\mathrm{Im}(g_\#) \approx Z + Z$. Case 2′ is when $\mathrm{Im}(g_\#) \approx \pi_1(K)$. We begin by showing that under certain hypotheses we have a positive answer.

Let us first recall some algebraic preliminaries about the fundamental group of the Klein bottle. Let us consider the presentation of $\pi_1(K)$, $\langle a, b | abab^{-1} \rangle$. We describe all subgroups $H \subset \pi_1(K)$ and the centralizer of a subgroup H of $\pi_1(K)$ denoted by $C_{\pi_1(K)}(H)$.

Lemma 4. *A subgroup H of $\pi_1(K)$ is either*

(a) *trivial*;
(b) *cyclic generated by an element of the form $a^r b^s$*;
(c) *either isomorphic to $Z+Z$ or to $\pi_1(K)$, if it is not generated by one element. In case* (c) *if the subgroup is isomorphic to $Z+Z$ then it is contained in the subgroup $\langle a, b^2\rangle$. Otherwise it will not be contained in the subgroup $\langle a, b^2\rangle$ and is of the form $\langle a^r, a^p b^{2q+1}\rangle$ where $r \neq 0$.*

Proof. This is a straightforward calculation. □

Now we describe the centralizer of these groups.

Lemma 5. *The centralizer of H is given as follows*:

(a′) *For H trivial $C_{\pi_1(K)}(H) = \pi_1(K)$.*
(b′) *For H as in* (b) *we have the following cases*:
  (I) *If $r = 0$ and s even then $C_{\pi_1(K)}(H) = \pi_1(K)$*;
  (II) *If $r = 0$ and s odd then $C_{\pi_1(K)}(H)$ is the subgroup $\langle b\rangle$*;
  (III) *If $r \neq 0$ and q is even then it is the subgroup $\langle a, b^2\rangle$*;
  (IV) *If $r \neq 0$ and s is odd then $C_{\pi_1(K)}(H) = \langle b^2\rangle$.*
(c′) *For H as in case* (c) *we have two cases*:
  (I) *If H is isomorphic to $Z+Z$ (so $H \subset \langle a, b^2\rangle$) then $C_{\pi_1(K)}(H) = \langle a, b^2\rangle$*;
  (II) *If H is isomorphic to $\pi_1(K)$ then $C_{\pi_1(K)}(H) = \langle b^2\rangle$.*

Proof. A straightforward calculation. □

We now present some positive results for maps into the Klein bottle.

Theorem 5.

(a) *Let $g : S \to K$, where S is an arbitrary closed surface. If the image of $g_\#$ contains an element of the form $a^r b$ with $r \neq 0$, then the restricted problem with respect to fixed homotopy at g is equivalent to the general coincidence problem. In particular this is the case if the image of $g_\#$ is all of $\pi_1(K)$ or it is the subgroup generated by $a^r b$ with $r \neq 0$.*
(b) *If S is the torus and the image of $g_\#$ equals the subgroup $< a, b^2 >$, then the restricted problem with respect to fixed homotopy at g is equivalent to the general coincidence problem.*

Proof. Let us first consider part (a). As a result of the hypothesis, Lemma 5 and Theorem 2, we have that $\pi_1(K^S, g)$ is isomorphic to the subgroup $\langle b^2\rangle$. But the induced homomorphism $\pi_1(K^K, id) \to \pi_1(K^S, g)$ is the identity under the identification given by Lemma 3. So the result follows from Theorem 1 since $\pi_1(K^K, id) \approx \langle b^2\rangle$.

For part (b) we consider the induced homomorphism $(g_*)_\# : \pi_1(T^T, id) \to \pi_1(K^T, g)$. From Lemma 2, part (b), we have that $\pi_1(T^T, id) = Z+Z$ and $\mathrm{Im}(g_\#)$ is $\langle a, b^2\rangle$. Arguing as in part (a) the result follows. □

In the next section we will consider the case where $\mathrm{Im}(g_\#)$ is the form $\langle b^s\rangle$.

4. Examples where the two problems are not equivalent

The purpose of this section is to provide some examples where the two problems are not equivalent. We will restrict our attention to the case where the target is the Klein bottle. In considering examples of maps from a surface into the Klein bottle our analysis will be done according to a classification of maps $l : S \to K$, similar to the one used in Section 3.2. Maps $l : S \to K$ are classified in terms of $\mathrm{Im}(l_\#)$: Case 0 is when $\mathrm{Im}(l_\#)$ is trivial; Case 1 is when $\mathrm{Im}(l_\#) \approx Z$; Case 2 is when $\mathrm{Im}(l_\#) \approx Z+Z$; Case 2′ is when $\mathrm{Im}(l_\#) \approx \pi_1(K)$. Recall that as a result of Theorem 5, in order to get examples where the two problems are not the same, we must assume that the image of $f_{1\#} : \pi_1(S) \to \pi_1(K)$ is not equal to $\pi_1(K)$. So the image is either trivial, isomorphic to Z, isomorphic to $Z+Z$ or a proper subgroup of $\pi_1(K)$ isomorphic to $\pi_1(K)$. This information will be useful to construct examples in each case.

We introduce the following notation: By an Example $SK\{ji\}$ for $i, j \in \{0, 1, 2, 2'\}$, we mean a pair of maps from a surface S to the Klein bottle K, such that the second coordinate g satisfies $\mathrm{Im}(g_\#)$ is as in Case i above and the first coordinate f satisfies $\mathrm{Im}(f_\#)$ is as in Case j.

We construct examples according to the classification of $\mathrm{Im}(g_\#)$, where g is a map which corresponds to the second coordinate. In Case I we will have $\mathrm{Im}(g_\#)$ trivial. For the remaining cases we will restrict ourselves to maps with domain either T or K, and these remaining cases, Case II, Case III, Case IV, are when $\mathrm{Im}(g_\#)$ is Case 1, 2, 2′, respectively.

Since the domain has Euler characteristic $\leqslant 0$, the number of generators (we can choose a canonical set of generators) is at least two. For the examples constructed below they satisfy $f_{1\#}(\rho_i) = f_{2\#}(\rho_i) = g_\#(\rho_i) = 1$ for $i > 2$. Also if S is nonorientable, then we define $f_{1\#}(\rho_1) = f_{1\#}(\rho_2)^{-1}$, $f_{2\#}(\rho_1) = f_{2\#}(\rho_2)^{-1}$, $g_\#(\rho_1) = g_\#(\rho_2)^{-1}$. If S is orientable, then $f_{1\#}(\rho_2) = f_{2\#}(\rho_2) = g_\#(\rho_2) = 1$. Thus in order to know the maps $f_{1\#}$, $f_{2\#}$ and $g_\#$, it suffices to prescribe the data $f_{1\#}(\rho_1)$, $f_{2\#}(\rho_1)$ and $g_\#(\rho_1)$.

Case I: Assume $\mathrm{Im}(g_\#)$ is trivial.

Example $SK\{00\}$ $\chi(S) \leqslant 0$. Let g be the constant map, so $g_\#(\rho_1) = 1$. Consider models for f_1, f_2 such that $f_{1\#}(\rho_1) = B$, $f_{2\#}(\rho_1) = \beta_1^{-1}B^{-1}\beta_1$. The two homomorphisms $(f_1 \times c)_\#, (f_2 \times c)_\# : \pi_1(S) \to P_2(K)$ are conjugate. This follows from the fact that the braids $B, \beta_1^{-1}B^{-1}\beta_1$ are conjugate. More precisely, $\beta_2 B \beta_2^{-1} = \beta_1^{-1}B^{-1}\beta_1$ from Section 2. Hence, (f_1, c) and (f_2, c) can be connected by a Wecken homotopy. But in the free group $F(\alpha_1, \beta_1)$ the two words B and $\beta_1^{-1}B^{-1}\beta_1$ are not conjugate, since they project to different elements on the abelianized. Therefore these two pairs cannot be connected by a Wecken homotopy where the second homotopy is the constant homotopy of the constant map, which establishes the example.

Example $SK\{10\}$ $\chi(S) \leqslant 0$. Here we construct an example similar to the one given in [9]. Define $f_{1\#}(\rho_1) = \alpha_1$, $f_{2\#}(\rho_1) = \alpha_1 B^{-1}$. Then the two homomorphisms $(f_1 \times c)_\#, (f_2 \times c)_\# : \pi_1(S) \to P_2(K)$ are conjugate as the braids $\alpha_1, \alpha_1 B^{-1}$ are conjugate. More precisely, $\beta_2^{-1}\alpha_1\beta_2 = \alpha_1 B^{-1}$ from Section 2. So the pair (f_1, c) (f_2, c) can be connected by a Wecken homotopy. But the two models f_1, f_2 cannot be connected by a root free homotopy since the two words α_1 and $\alpha_1 B^{-1}$ are certainly not conjugate as they project to different elements in the abelianized of the free group on 2 letters.

Examples of the form Example $SK\{20\}$ or Example $SK\{2'0\}$ do not exist because if $\mathrm{Im}(f_\#)$ has finite index in $\pi_1(K)$, then necessarily g is not root free. This now completes Case I.

Case II: Suppose $\mathrm{Im}(g_\#)$ is of rank 1.

Example $SK\{01\}$ $\chi(S) \leqslant 0$. This example illustrates the case where the rank of $\mathrm{Im}(g_\#)$ is one, while the other map is homotopic to the constant map. Let g be a map such that $g_\#(\rho_1) = \beta^2$. And thus $g_\#(\rho_2) = \beta^{-2}$, $g_\#(\rho_i) = 1$, $i > 2$. Let f_1 be the constant map so $f_{1\#}(\rho_j) = 1$ for all j. Instead of directly giving a model for the map f_2 we will create one in the process below. First let us assume that $S = N_l$ is a nonorientable surface. The induced homomorphism $\pi_1(N_l) \to \pi_1(K \times K - \Delta)$ by the pair (f_1, g) is given by $\rho_1 \to \beta_2^2$, $\rho_2 \to \beta_2^{-2}$ and $\rho_i \to 1$ for $i > 2$. Take the conjugate of this homomorphism by the element α_2.

This new homomorphism is given by

$$\rho_1 \to \alpha_2\big(\beta_2^2\big)\alpha_2^{-1} = \big(\alpha_2\beta_2^2\alpha_2^{-1}\beta_2^{-2}\big)\beta_2^2 \quad \text{and}$$
$$\rho_2 \to \alpha_2\big(\beta_2^{-2}\big)\alpha_2^{-1} = \big(\alpha_2\beta_2^{-2}\alpha_2^{-1}\beta_2^2\big)\beta_2^{-2}.$$

Now we will compute the element $\alpha_2\beta_2^{-2}\alpha_2^{-1}\beta_2^2$, which is an element of the kernel of $\pi_1(K \times K - \Delta) \to \pi_1(K \times K)$. We get

$$\begin{aligned}\alpha_2\beta_2^2\alpha_2^{-1}\beta_2^{-2} &= \big(\alpha_2\beta_2\alpha_2\beta_2^{-1}\big)\beta_2\big(\alpha_2^{-1}\big(\beta_2\alpha_2^{-1}\beta_2^{-1}\alpha_2^{-1}\big)\alpha_2\big)\beta_2^{-1}\\ &= B\beta_2\big(\alpha_2^{-1}B^{-1}\alpha_2\big)\beta_2^{-1} = B\beta_1^{-1}\alpha_1^{-1}B\alpha_1\beta_1.\end{aligned}$$

Now let $f_2 : N_l \to K$ be a map such that defined on the one skeleton it sends the loop ρ_1 to $B\beta_1^{-1}\alpha_1^{-1}B\alpha_1\beta_1$ and it sends the loop ρ_2 to $(\alpha_2\beta_2^{-2}\alpha_2^{-1}\beta_2^2)$ whatever this may be.

The two pairs can be connected by a Wecken homotopy since the induced homomorphisms are conjugate. It remains to show that there is no element w of the subgroup generated by α_1, β_1 such that $w\beta_2^2w^{-1} = B\beta_1^{-1}\alpha_1^{-1}B\alpha_1\beta_1\beta^2$ or equivalently, $w\beta_2^2w^{-1}\beta_2^{-2} = B\beta_1^{-1}\alpha_1^{-1}B\alpha_1\beta_1$.

This equation, when viewed on the abelianized of the free group $F(\alpha_1, \beta_1)$ becomes $w\beta_2^2w^{-1}\beta_2^{-2} = B^2$. The conjugation by β_2^2 on the abelianized of $F(\alpha_1, \beta_1)$, from our table in Section 2, is the identity. So our equation on the abelianized becomes $B^2 = 1$ or $\alpha_1^2\alpha_1^2 = 1$ from the relations in Section 2, which is a contradiction and the result follows. The case when S is orientable is completely analogous and is left to the reader to verify.

Example $SK\{11\}$ $\chi(S) \leqslant 0$. In this example the rank of the image of the induced homomorphisms for both homotopy classes $[f], [g]$ is one. Let us first consider the case where $S = N_l$ is a nonorientable surface. Consider homotopy classes of maps $[f], [g]$ such that if $f \in [f]$ and $g \in [g]$, then $f_\#(\rho_1) = \beta^2$, $f_\#(\rho_2) = (\beta)^{-2}$, and $g_\#(\rho_1) = \beta^2$, $g_\#(\rho_2) = (\beta)^{-2}$.

Consider the homomorphism $\pi_1(N_l) \to \pi_1(K \times K - \Delta)$ which sends ρ_1 to $\beta_1^2\beta_2^2$. As in the cases before ρ_2 is sent to $(\beta_1^2\beta_2^2)^{-1} = \beta_2^{-2}\beta_1^{-2} = \beta_2^{-2}\beta_1^{-2}\beta_1^2\beta_1^{-2}$.

But it suffices to analyze what happens with the images of ρ_1. As in the previous example we take the conjugate of this element by β_2. Then we obtain an element w and consequently a homomorphism which send ρ_1 to w, ρ_2 to w^{-1} and all other generators to 1. Consider a map which induces this homomorphism. Using the braid relations we compute w which is

$$w = (\beta_2)\left(\beta_1^2\beta_2^2\right)(\beta_2)^{-1} = \left(\beta_1^{-1}B\beta_1^2\right)\left(\beta_1^{-1}B\beta_1^2\right)\beta_2^2.$$

We claim that the two elements $\beta_1^2\beta_2^2$ and w are not conjugate by some element of the first coordinate. Suppose that there is a θ such that $\theta\beta_1^2\beta_2^2\theta^{-1} = (\beta_1^{-1}B\beta_1^2)(\beta_1^{-1}B\beta_1^2)\beta_2^2$. This is equivalent to showing that:

$$\theta\beta_1^2\beta_2^2\theta^{-1}\beta_2^{-2} = \left(\beta_1^{-1}B\beta_1^2\right)\left(\beta_1^{-1}B\beta_1^2\right).$$

We now follow the same reasoning as in the previous example. We project on the abelianized, and use fact that the action of β_2^2 on the abelianized is trivial. So the equation on the abelianized becomes

$$\theta\beta_{11}^2\theta^{-1} = \beta_1^{-1}B\beta_1B\beta_1^2 \quad \text{or} \quad 1 = B^2,$$

which is a contradiction and the result follows. The case when S is orientable is completely analogous and is left to the reader to verify.

To complete the remaining two possibilities of Case II, from now on we assume the domain is either the torus T or the Klein bottle K. We will use the following two elementary facts. If $g: K \to K$ is a map, then $\mathrm{Im}(g_\#)$ is not isomorphic to $Z + Z$, since the abelianized $(\pi_1(K))_{ab}$ has rank 1. If $g: T \to K$ is a map, then $\mathrm{Im}(g_\#)$ is not isomorphic to $\pi_1(K)$ since the image is abelian.

Under the assumptions above let us consider potential examples where $\mathrm{Im}(g_\#) \approx Z$ and we have the following possibilities: $KK\{2'1\}$, $KK\{21\}$, $TK\{2'1\}$, $TK\{21\}$.

We observe that the cases Example $KK\{21\}$ and Example $TK\{2'1\}$ do not exist as a result of the comments just above.

The following example deals with the case not covered in Theorem 5.

Example $KK\{2'1\}$. We use the presentation of $P_2(K)$ given in terms of α, β at the end of Section 2.

Define $f_{1\#}(\alpha) = \alpha$, $f_{1\#}(\beta) = \beta$; $g_\#(\alpha) = 1$, $g_\#(\beta) = \beta$. So we have that $(f_1, g)_\#(\alpha) = \alpha_1$, $(f_1, g)_\#(\beta) = \beta_1\beta_2$.

Consider the homomorphism such that $(f_2, g)_\#(\alpha) = \beta_2^2\alpha_1\beta_2^{-2} = \beta_1^{-2}B^{-1}\alpha_1B\beta_1^2$, and $(f_2, g)_\#(\beta) = \beta_2^2\beta_1\beta_2\beta_2^{-2} = \beta_1^{-2}B^{-1}\beta_1B\beta_1^2\beta_2$.

That is, the second homomorphism corresponds to the first homomorphism conjugated by the element β_2^2. Hence, there exists a Wecken homotopy between the corresponding pairs.

We claim that there is no element $w \in P_2(K)$ such that the projection on the second coordinate is β, and such that $w\beta_2^{-2}$ commutes with α_1, which is in the image $\mathrm{Im}(f_1, g)_\#$. By Proposition 4, this is equivalent to saying that there is no Wecken homotopy such that in the second coordinate we have a homotopy which correspond to the element β_2.

To show the algebraic fact above, using the presentation of $P_2(K)$ it follows that conjugation by α_2 on the abelianized of the subgroup generated by $\{\alpha_1, \beta_1\}$ is the identity. Conjugation by β_2 maps $\alpha_1 \to \alpha^{-1}$. Thus, in the abelianized conjugation by $w\beta_2^{-2}$ sends α_1 to α_1^{-1}, as the exponent of β_2 in $w\beta_2^{-2}$ is odd. But this cannot occur if the words $w\beta_2^{-2}$ and α_1 commute, so the results follows.

Example $TK\{21\}$. This is similar to the previous example.

Let a, b be generators for $\pi_1(T)$ which project to α, β^2.

Define

$$f_{1\#}(a) = \alpha, \qquad f_{1\#}(b) = \beta^2; \qquad g_\#(a) = 1, \qquad g_\#(b) = \beta^2.$$

Choose a model f_1 such that $(f_1, g)_\#(a) = \alpha_1$ and $(f_1, g)_\#(b) = B\beta_1^2\beta_2^2$. Take conjugation by β_2^2 to get a new pair (f_2, g).

Now the argument proceeds as in the previous example, i.e. there is no Wecken homotopy such that on the second coordinate it is a homotopy G which corresponds to the element β_2.

To conclude this section we will comment on Cases III and IV.

Case III: Suppose $\mathrm{Im}(g_\#) \approx Z + Z$.

In general we have the following possibilities: $KK\{2'2\}$, $KK\{22\}$, $KK\{12\}$, $KK\{02\}$ and $TK\{2'2\}$, $TK\{22\}$, $TK\{12\}$, $TK\{02\}$. But the cases with domain K do not exist since $\mathrm{Im}(g_\#)$ cannot be $Z + Z$ as pointed out at the beginning of the section. Similarly, $TK\{2'2\}$ is not possible due to the fact that $\mathrm{im}(f_{i\#})$ is abelian as mentioned above. The case $TK\{02\}$ cannot happen because the map g cannot be deformed to be root free. So we are left with 2 possible cases $TK\{22\}$ and $TK\{12\}$. Further, in the former case from Theorem 5 in Section 3.2, we have that the only possible examples are in the case where the image of $g_\#$ is a proper subgroup of rank 2 of $\langle\alpha, \beta^2\rangle \subset \pi_1(K)$.

Case IV: Let $\mathrm{Im}(g_\#) \approx \pi_1(K)$.

In general we have the same eight possibilities as in Case III. But the cases with domain T do not exist since $\mathrm{Im}(g_\#)$ cannot be isomorphic to $\pi_1(K)$ as pointed out at the beginning of the section. Similarly, $KK\{22'\}$ is not possible due to the fact that $\mathrm{Im}(f_{i\#})$ is not isomorphic to $Z+Z$ as mentioned above. The case $KK\{02'\}$ cannot happen because the map g cannot be deformed to be root free. So we are left with 2 possible cases $KK\{2'2'\}$ and $KK\{12'\}$.

The study of the 4 potential types of examples mentioned above is still under progress and we do not know the answer at the moment.

5. Maps from a surface into $\mathbb{R}P^2$

Recall that for maps from a surface into the 2-sphere, the problem has been solved. This is Corollary 1.3 from [7] which states that the two questions are equivalent. The purpose of this section is to do the analysis when the target surface has positive Euler characteristic, and so we assume the target surface is $\mathbb{R}P^2$. In this section we give a classification of maps and use this to characterize the pairs of maps that are coincidence free. Proposition 4 is then applied to decide when the two Wecken problems are the same or different for a given pair of coincidence free maps.

We begin by recalling the classification of the homotopy classes of maps $[S, S^2]$ (base point free, which is the same as base point preserving) from a surface S into S^2, as well as the homotopy classification of maps $[S, \mathbb{R}P^2]$ (base point free).

Proposition 5. *The set of maps $[S, S^2]$ are classified as follows*:

(a) *If S is orientable, then it is identified with Z and the correspondence is given by the degree of the map.*
(b) *If S is nonorientable, then it contains exactly two elements.*

Proof. This follows from the Hopf classification theorem where the set is in one-to-one correspondence with $H^2(S, Z)$. □

For the classification of the homotopy classes of maps $[S, \mathbb{R}P^2]$ (base point free maps) from a surface into the projective space $\mathbb{R}P^2$, which is the set of path components of $(\mathbb{R}P^2)^S$, we use [12]. There one denotes by $m(S, \mathbb{R}P^2)$ the set $(\mathbb{R}P^2)^S$, and by $m(S, \mathbb{R}P^2; f)$ the subset of the function space $(\mathbb{R}P^2)^S$ which consist of the maps which are homotopic to f.

Proposition 6. *The set of components of $(\mathbb{R}P^2)^S$ are described as follows.*

If S is orientable;

$$m(S, \mathbb{R}P^2) = \coprod_{k=0}^{\infty} m(S, \mathbb{R}P^2; f_{2k}) \sqcup \coprod_{x \in H^1(S;Z/2),\, x \neq 0} \big(m(S, \mathbb{R}P^2; f_x) \sqcup m(S, \mathbb{R}P^2; f_x^-)\big),$$

where the homomorphism induced on the fundamental group and the corresponding class in H^1 are

$$(f_{2k})_* = 0,$$
$$(f_x^{\pm})_* = x \neq 0,$$

and the twisted degree is

$$d(f_{2k}) = 2k, \qquad d(f_x^{\pm}) = 0.$$

If S is nonorientable;

$$m(S, \mathbb{R}P^2) = \coprod_{k=0}^{\infty} m(S, \mathbb{R}P^2; f_{2k+1}) \sqcup \coprod_{x \in H^1(S;Z/2),\, x \neq w_1(\tau S),\, x^2=1} m(S, \mathbb{R}P^2; f_x^1)$$
$$\sqcup \coprod_{x \in H^1(S;Z/2),\, x \neq w_1(\tau S),\, x^2=0} \big(m(S, \mathbb{R}P^2; f_x^0) \sqcup m(S, \mathbb{R}P^2; f_x^{0-})\big),$$

where

$$(f_{2k+1})_* = w_1(\tau S),$$
$$(f_x^{i\pm})_* \neq w_1(\tau S), \qquad x \neq w_1(\tau S),$$

and

$$d(f_{2k+1}) = 2k+1, \qquad d(f_x^{i\pm}) = i.$$

In the proposition above we identify $\mathrm{Hom}(\pi_1(S), Z_2) = \mathrm{Hom}(H_1(S), Z_2) = H^1(S, Z_2)$.

Now we classify the pairs of homotopy classes of maps $([f],[g])$ for which the minimal number of coincidence points is zero, i.e. there are $f \in [f]$, $g \in [g]$ such that (f,g) is coincidence free. For each map we have the induced homomorphism on the fundamental group which is a homomorphism $\pi_1(S) \to \pi_1(\mathbb{R}P^2) = Z_2$. We will use the presentation $\langle a_1, b_1, \dots, a_g, b_g | [a_1, b_1] \dots [a_g, b_g] \rangle$ for the orientable closed surface of genus g and $\langle \rho_1, \dots, \rho_g | \rho_1^2 \dots \rho_g^2 \rangle$ for the closed nonorientable surface of genus g. Some aspects of coincidence sets of maps from a space into projective space has been studied in [16]. See also in [6] some related material. The result that we need is more in the spirit of the Wecken property and is partially related to some results of [16]. For the notion of absolute degree which is going to be used, see [2].

Proposition 7. *Let $f, g : S \to \mathbb{R}P^2$ be a pair of maps.*

(i) *If (f, g) can be made coincidence free, then both maps have absolute degree zero.*
(ii) *If f has absolute degree zero and its induced homomorphism on the fundamental group is trivial, then f is null homotopic.*

Proof. It is well known that the pair (f,g) can be made coincidence free if and only if the map $f, g : S \to \mathbb{R}P^2 \times \mathbb{R}P^2$ factors through, up to homotopy, the subspace $\mathbb{R}P^2 \times \mathbb{R}P^2 - \Delta$. The space $\mathbb{R}P^2 \times \mathbb{R}P^2 - \Delta$ is the 2nd-configuration space on $\mathbb{R}P^2$. From [20] this space has fundamental group isomorphic to the quaternionic Q_8 and from [5], Proposition 6, its universal covering has the homotopy type of the 3-sphere. Let $h : S \to \mathbb{R}P^2 \times \mathbb{R}P^2 - \Delta$ be a homotopy lifting of (f,g) and $p_i : \mathbb{R}P^2 \times \mathbb{R}P^2 - \Delta \to \mathbb{R}P^2$ the projections for $i = 1, 2$. The composites $p_1 \circ h$, $p_2 \circ h$, are homotopic to f, g, respectively. We claim that they have absolute degree zero. We divide the proof into several cases because of the definition of the absolute degree. In order to show that f has absolute degree zero, let $(\mathbb{R}P^2 \times \mathbb{R}P^2 - \Delta)^* \to \mathbb{R}P^2 \times \mathbb{R}P^2 - \Delta$ be the finite covering which corresponds to the subgroup which is the kernel of the homomorphism $p_{1\#} : \pi_1(\mathbb{R}P^2 \times \mathbb{R}P^2 - \Delta) \to \pi_1(\mathbb{R}P^2)$. So there is a lifting $\hat{p}_1 : (\mathbb{R}P^2 \times \mathbb{R}P^2 - \Delta)^* \to S^2$ of p_1. Then we have the commutative diagram:

$$\begin{tikzcd}
(\mathbb{R}P^2 \times \mathbb{R}P^2 - \Delta)^* \arrow[r, "\hat{p}_1"] \arrow[d] & S^2 \arrow[d] \\
\mathbb{R}P^2 \times \mathbb{R}P^2 - \Delta \arrow[r] & \mathbb{R}P^2
\end{tikzcd}$$

We now suppose that S is orientable. If the induced homomorphism $(p_1 \circ f)_\# : \pi_1(S) \to \pi_1(\mathbb{R}P^2)$ is trivial, then the map lifts into the covering S^2 of $\mathbb{R}P^2$ and the absolute degree is the degree of the lifting from S into S^2. But this map factors through $(\mathbb{R}P^2 \times \mathbb{R}P^2 - \Delta)^* \to \mathbb{R}P^2$. Because $H_2((\mathbb{R}P^2 \times \mathbb{R}P^2 - \Delta)^*; Z) \approx H_2(\pi_1(\mathbb{R}P^2 \times \mathbb{R}P^2 - \Delta)^*; Z) = H_2(Z_4; Z) = 0$ is trivial it follows that the degree is zero. If the induced homomorphism $(p_1 \circ f)_\# : \pi_1(S) \to \pi_1(\mathbb{R}P^2)$ is not trivial (so surjective) we consider the double covering of S which corresponds to the subgroup which is the kernel of the homomorphism $(p_1 \circ f)_\#$. From now on we follow the same steps as in the previous case.

The case where S is nonorientable is similar and we leave the details to the reader. This completes the proof of the part (i).

For the second part, the map f lifts to a map $\hat{f} : S \to S^2$. Because the absolute degree is zero, $\hat{f}$ is null homotopic and the result follows. □

The proposition above together with the classification of the homotopy classes of maps from a closed surface into $\mathbb{R}P^2$, tell us that we have only a finite number of pairs of homotopy classes $([f],[g])$ such that the minimal number of coincidence points is zero.

Proposition 8. *There is a bijection between the set of homotopy classes of maps $[S, \mathbb{R}P^2 \times \mathbb{R}P^2 - \Delta]$ (base point free) and the set of conjugacy classes of homomorphisms $\mathrm{Hom}(\pi_1(S), Q_8)$.*

Proof. From classical obstruction theory given X a CW-complex of dimension n and Y a space there is a bijection between the set of homotopy classes of maps $[X, Y]$ and $[X, Y_n]$, where Y_n is the nth-stage of the Postnikov system of Y. In our case the set $[S, \mathbb{R}P^2 \times \mathbb{R}P^2 - \Delta]$ is in one-to-one correspondence with $[S, (\mathbb{R}P^2 \times \mathbb{R}P^2 - \Delta)_2]$. But $(\mathbb{R}P^2 \times \mathbb{R}P^2 - \Delta)_2 = K(Q_8, 1)$ as a result of [5], Proposition 6. From [22], Chapter V, Section 4, Theorem 4.3, the result follows. □

Now we characterize all pairs $([f],[g])$ such that the minimal number of coincidence is zero. To do so we introduce the following notation:

(I) Let S_g be the orientable surface of genus g, $f, g : S_g \to \mathbb{R}P^2$ and $(f,g) : S_g \to \mathbb{R}P^2 \times \mathbb{R}P^2$. If $f_\#, g_\# : \pi_1(S_g) \to Z_2$ are the induced homomorphisms by f, g on the fundamental group, denote by $\mathrm{Im}_i(f,g)$ the image of the homomorphism $(f_\#, g_\#) : \pi_1(S_g) \to \pi_1(\mathbb{R}P^2 \times \mathbb{R}P^2) \cong Z_2 + Z_2$ restricted to the subgroup generated by a_i, b_i.

(II) Let N_g be the nonorientable surface of genus g, $f, g : N_g \to \mathbb{R}P^2$ and $(f,g) : S_g \to \mathbb{R}P^2 \times \mathbb{R}P^2$. If $f_\#, g_\# : \pi_1(N_g) \to Z_2$ are the induced homomorphisms by f, g on the fundamental group, denote by $\mathrm{Im}_i(f,g)$ the image of the homomorphism $(f_\#, g_\#) : \pi_1(N_g) \to \pi_1(\mathbb{R}P^2 \times \mathbb{R}P^2) \cong Z_2 + Z_2$ restricted to the subgroup generated by ρ_i.

Proposition 9. *Suppose both f and g have absolute degree zero. The pair (f,g) can be deformed to a coincidence free pair if and only if*:

(a) *In the case the surface is orientable S_g, the number of integers i such that $\mathrm{Im}_i(f,g) = Z_2 + Z_2$ is even.*
(b) *In the case the surface is nonorientable N_g, the number of integers i such that $\mathrm{Im}_i(f,g) = Z_2$ is even.*

Proof. From Proposition 8 it suffices to show that the algebraic diagram has a lift. If $\mathrm{Im}_i(f,g)$ has cardinality $\leqslant 2$, then the preimage of this subgroup under the homomorphism $\pi_1(\mathbb{R}P^2 \times \mathbb{R}P^2 - \Delta) = Q_8 \to Z_2 + Z_2$ is either Z_2 or Z_4. In both cases we can define a lifting for a_i, b_i such that the commutator of the image of these elements are trivial. If $\mathrm{Im}_i(f,g) = Z_2 + Z_2$, a simple analysis of all possible lifting shows that the commutator of the lifting of the two elements is the element $-1 \in Q_8$. In order to get a homomorphism, the relation $[a_1,b_1]\dots[a_g,b_g]$ should be preserved, which is equivalent to having an even number of values of i such that $\mathrm{Im}_i(f,g) = Z_2 + Z_2$. The converse is clear and part (a) follows. The second part is similar and we leave to the reader. $\square$

Now, by means of Proposition 4, the comparison of the general problem and the restricted problem becomes an algebraic problem about homomorphisms into $Q_8 = \pi_1(\mathbb{R}P^2 \times \mathbb{R}P^2 - \Delta)$. As before we consider $(f_1,g), (f_2,g)$ a pair of coincidence free maps. Recall that if a pair (f,g) is coincidence free, then it defines a map also denoted by $(f,g): S \to \mathbb{R}P^2 \times \mathbb{R}P^2 - \Delta$. When we consider this map as a map $(f,g): S \to \mathbb{R}P^2 \times \mathbb{R}P^2$, then we denoted by $(f_\#, g_\#)$ the induced homomorphism on the fundamental group $(f_\#, g_\#): \pi_1(S) \to \pi_1(\mathbb{R}P^2 \times \mathbb{R}P^2)$. We state our main result of the section in terms of the image of the homomorphism $(f_\#, g_\#)$, where $[f] = [f_i]$, $i = 1, 2$. For instance, with f, f_1, f_2, g as above we will show that if $\mathrm{Im}(f_\#, g_\#)$ is the trivial subgroup, then the two homomorphisms $(f_1,g)_\#, (f_2,g)_\#$ are conjugate if and only if they are equal. So it then follows that the two problems are equivalent. The complete picture is obtained by consideration of the cases when $\mathrm{Im}(f_\#, g_\#)$ is isomorphic to Z_2, and when $\mathrm{Im}(f_\#, g_\#)$ is isomorphic to $Z_2 + Z_2$.

The relation between the two problems when the target surface is $\mathbb{R}P^2$ is given in the following theorem. Let $Z_2 + 0 \subset \pi_1(\mathbb{R}P^2 \times \mathbb{R}P^2) \cong Z_2 + Z_2$ the first summand. and $\iota: \mathbb{R}P^2 \times \mathbb{R}P^2 - \Delta \to \mathbb{R}P^2 \times \mathbb{R}P^2$ the inclusion.

Theorem 6. *For maps from a closed surface into $\mathbb{R}P^2$ the general problem and the restricted problem are related as follows*:

(I) *If $\mathrm{Im}(f_\#, g_\#)$ is trivial, then the two problems are equivalent.*
(II) *If $\mathrm{Im}(f_\#, g_\#)$ is $Z_2 + 0$, then the two problems are different if and only if the two homomorphisms $(f_1,g)_\#, (f_2,g)_\#$ are different. If $\mathrm{Im}(f_\#, g_\#)$ is either $0 + Z_2$ or the diagonal $\Delta \subset Z_2 + Z_2$, then the two problems are the same.*
(III) *If $\mathrm{Im}(f_\#, g_\#)$ is isomorphic to $Z_2 + Z_2$, the two problems are the same if and only if the two homomorphisms $(f_1,g)_\#$ and $(f_2,g)_\#$ are conjugate by an element of the subgroup $\iota_\#^{-1}(Z_2 + 0)$.*

Proof. Part (I). $\mathrm{Im}(f_\#, g_\#)$ being trivial implies that $\mathrm{Im}(f_1,g)_\#$ and $\mathrm{Im}(f_2,g)_\#$ are contained in the center of $\pi_1(\mathbb{R}P^2 \times \mathbb{R}P^2 - \Delta) = Q_8$, which is isomorphic to Z_2. Assuming there is a Wecken homotopy connecting the two pairs the two homomorphisms are conjugate, and it then follows that they must be equal. Therefore conjugation by any element of one homomorphism gives the other, and the result follows by Proposition 4.

In order to simplify the proof of the remaining parts let us consider the usual description of Q_8 given by the elements $\{\pm i, \pm j, \pm k\}$, and without loss of generality, that the homomorphism on the fundamental group induced by the inclusion $\iota: \mathbb{R}P^2 \times \mathbb{R}P^2 - \Delta \to \mathbb{R}P^2 \times \mathbb{R}P^2$ is given by $i \to (\bar{1}, \bar{0})$, $j \to (\bar{0}, \bar{1})$, $k \to (\bar{1}, \bar{1})$.

Part (II). Assuming $\mathrm{Im}(f_\#, g_\#)$ is $Z_2 + 0$, it follows that $\mathrm{Im}(f_1,g)_\#$ and $\mathrm{Im}(f_2,g)_\#$ are contained in the subgroup generated by i, and the two homomorphisms are either equal or conjugates by either j or k. If they are equal, then the two problems are the same. Otherwise, the two homomorphisms cannot be conjugated by an element which projects to $(\bar{1}, \bar{0})$. So the first part follows. The second part is similar. The image of the two homomorphisms are either the subgroup generated by j or by k. The homomorphisms are then conjugated by the trivial element or by i, and so the conclusion follows.

Part (III). Assuming $\mathrm{Im}(f_\#, g_\#)$ is $Z_2 + Z_2$, it follows that $\mathrm{Im}(f_1,g)_\#$ and $\mathrm{Im}(f_2,g)_\#$ are equal to Q_8. Now suppose that the two problems are equivalent. The two homomorphisms $(f_1,g)_\#$ and $(f_2,g)_\#$ are conjugate because the two maps (f_1,g), (f_2,g) are homotopic. It remains to show that they are conjugate by an element of the subgroup generated by $i \in Q_8$. Suppose that they are conjugated by either $\pm j$ or $\pm k$. The conjugation by any one of these elements send i to $-i$. But these two homomorphisms should also be conjugated by an element of $\iota_\#^{-1}(Z_2 + 0)$ because the two problems are equivalent. But this is a contradiction. So necessarily we have conjugation by an element of $\iota_\#^{-1}(Z_2 + 0)$. For the converse, we have that the two homomorphisms are conjugate by an element of the subgroup generated by i. So they are conjugated by either ± 1 or $\pm i$. In all cases the conjugation restricted to the subgroup generated by i is the identity. So the only coincidence Wecken homotopies which are allowed, are the ones where the second coordinate is the constant homotopy and the result follows. $\square$

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